ECE 3100 Probability and Inference / Prelim 1 Cheatsheet / Will Barkoff wb273@cornell.edu / Tuesday, March 8, 2022

Basic Definitions

A **probabalistic model** is a mathematical desc of a random experiment.
Probability theory allows us to predict and make sense of data.
A **random experiment** does not have a predetermined outcome and repeating it may lead to different outcomes.
The **sample space**, denoted Ω, is the set of all outcomes.
It must be **mutually exclusive** (or disjoint) and **exhaustive**.
A sample space is **discrete** if it's a countable size.
It is **continuous** if it is uncountable.
An **event** is a subset of the sample space. The event Ω always occurs. The event $\emptyset$ never occurs.

Set Theory

Set operations:

union	$A \cup B$	things in A and things in B	"or"
intersection	$A \cap B$	things in both A and B	"and"
complement	A^c	things in Ω but not A	"not"

Properties of set operations

Commutivity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$	$(A \cap B) \cap C = A \cap (B \cap C)$
De Morgans	$(A \cup B)^c = A^c \cap B^c$	$(A \cap B)^c = A^c \cup B^c$

Set relations

$A \subseteq B \Leftrightarrow A \cap B = A$ "subset"
$A \supseteq B \Leftrightarrow A \cap B = B$ "superset"
$A \cap B = \emptyset \Leftrightarrow$ "disjoint"

Probabalistic Models

We define the **probabalistic model** of a random experiment to be $(\Omega, \mathcal{F}, \Pr)$

Ω is the sample space, $\mathcal{F}$ is a collection of events, and $\Pr: \Omega \to \mathbb{R}$ is a **probability law**, which must adhere to the following axioms:

- **Nonnegativity**: $\Pr(A) \geq 0$
- **Normalization** $\Pr(\Omega) = 1$
- **Additivity**: $A \cap B = \emptyset \Rightarrow \Pr(A \cup B) = \Pr(A) + \Pr(B)$

Using these axioms, we derive:

$\Pr(A^c) = 1 - \Pr(A)$ $\quad \Pr(\emptyset) = 0$ $\quad \Pr(\Omega) = 1$

$A \subseteq B \Rightarrow \Pr(A) \leq \Pr(B)$ $\quad A \subseteq B \Rightarrow \Pr(A) \leq 1$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

For countable disjoint sets:

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(A_i)$$

For a finite partition: Disjoint $\{S_i\}_i$ and $\Omega = \bigcup_{i=1}^{n} S_i$

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A \cap S_i)$$

Union bound: $\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$

Calculating Probability

Discrete uniform law: If every event in the sample space is equiprobable:

$$\Pr(A) = \frac{|A|}{|\Omega|}$$

Counting Principle: If an experiment has r stages with n_i choices stage i, the total number of choices is $n_1 n_2 \cdots n_r$

Number of Subsets of A: $2^{|A|}$ $\quad$ **K-permutations**: $\frac{n!}{(n-k)!}$

Combinations: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Continuous Sample Spaces: use portions of area

Conditional Probability

The conditional probability of event A given that event B has occured is defined to be: $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

Multiplication Rule: $\Pr(A \cap B) = \Pr(B)\Pr(A|B)$

Total Probability Theorem: Given $\{A_1, \dots, A_n\}$ partitions the sample space:

$$\Pr(B) = \Pr(A_1)\Pr(B|A_1) + \dots + \Pr(A_n)\Pr(B|A_n)$$

Bayes' Theorem:

$$\Pr(B|A) = \frac{\Pr(A|B)\Pr(B)}{\Pr(A)}$$

Generalized Bayes' Theorem:

$$\Pr(A_i|B) = \frac{\Pr(A_i)\Pr(B|A_i)}{\sum_{j=1}^{n} \Pr(A_j)\Pr(B|A_j)}$$

Independence and Conditional Independence

Two events are **independent** if one event occuring doesn't impact the probability of the other. $\Pr(A \cap B) = \Pr(A)\Pr(B)$

Events $A_1, \dots, A_n$ are independent if $\Pr\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \Pr(A_i)$ for every subset of events S

Events A and B are **conditionally independent** given C if $\Pr(A \cap B | C) = \Pr(A|C)\Pr(B|C)$

Random Variables

A **random variable** is a mapping of the sample space to real numbers. $X: \Omega \to \mathbb{R}$

The **support** of a random variable is the possible values it can take. The **probability mass function** (PMF) maps a value to the probability that the random variable will take that value.

$p_X(x) \geq 0$ $\quad \sum_x p_X(x) = 1$ $\quad p_X(x) = \Pr(X = x) = \Pr(\{w : X(w) = x\})$ $w \in \Omega$

The **expectation** of a random variable:

$$\mathbb{E}[X] = \sum_x x\, p_X(x)$$

Linearity of Exp: $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

Expected Value Rule: $\mathbb{E}[g(X)] = \sum_x g(x)\, p_X(x)$

The **variance** of a random variable is defined to be

$$\text{var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Random Var	Desc	PMF(k)	E	Var
$X \sim Ber(p)$	1 w/ prob p $\quad q = 1-p$	0: q $\quad$ 1: p	p	pq
$X \sim B(n, p)$	n Ber trials, no. successes	$\binom{n}{k} p^k q^{n-k}$	np	npq
$X \sim Geo(p)$	# of 1st success on rep Ber	$p(1-p)^k$	$\frac{1}{p} - 1$	$\frac{1-p}{p^2}$
$X \sim Pois(\lambda)$	estimated $B(n,p)$ where $\lambda = np$	$\frac{e^{-\lambda}\lambda^k}{k!}$	λ	λ
$X \sim U(a, b)$	equal liklihood of any # between a, b	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{1}{12}(b-a)^2$

Bias Detection

Bob has 5 coins. One is biased, the others are fair. He picks one coin at random and tosses it. The probability that the biased coin lands heads is $p < 1/2$.

① Let H_1 be the event that he sees a head on the first toss. Compute $Pr(H_1)$.

Let B be the event that the biased coin is chosen. By the total probability theorem:

$$Pr(H_1) = Pr(B)Pr(H_1 | B) + Pr(B^c)Pr(H_1 | B^c)$$
$$= \frac{1}{5}p + \frac{4}{5}\cdot\frac{1}{2} = \frac{p+2}{5}$$

② What is the probability given H_1 that the coin is biased?

By Bayes' Theorem:

$$Pr(B|H_1) = \frac{Pr(H_1|B)Pr(B)}{Pr(H_1)} = \frac{\frac{p}{5}}{\frac{p+2}{5}} = \frac{p}{p+2}$$

③ Suppose that bob flips the coin again and it is heads. What is the new probability that the coin is biased.

Our updated $Pr(B)$ is $\frac{p}{p+2}$. We must now compute $Pr(H_2)$, which we can use the total prob thrm for:

$$Pr(H_2) = Pr(B)Pr(H_2|B) + Pr(B^c)Pr(H_2|B^c)$$
$$= \frac{p}{p+2}p + \left(1 - \frac{p}{p+2}\right)\frac{1}{2} = \frac{p^2}{p+2} - \frac{p}{2p+4} + \frac{1}{2}$$
$$= \frac{2p^2+p}{p+2} + \frac{1}{2} = \frac{p^2+1}{p+2}$$

Now we can again use Bayes' theorem to compute:

$$Pr(B|H_2) = \frac{Pr(H_2|B)Pr(B)}{Pr(H_2)} = \frac{\frac{p^2}{p+2}}{\frac{p^2+1}{p+1}} = \frac{p^2}{p^2+1}$$

Digital Communications

Consider a digital communications channel outlined below:

Input: X Output: Y $a \to \alpha$ $b \to \beta$ $c \to \gamma$

$Pr(X=a) = 0.3$ $Pr(X=b) = 0.5$ $Pr(X=c) = 0.2$

$Pr(Y=\alpha|X=a) = 0.6$ $Pr(Y=\alpha|X=b) = 0.1$ $Pr(Y=\alpha|X=c) = 0.1$

$Pr(Y=\beta|X=a) = 0.3$ $Pr(Y=\beta|X=b) = 0.5$ $Pr(Y=\beta|X=c) = 0.1$

$Pr(Y=\gamma|X=a) = 0.8$ $Pr(Y=\gamma|X=b) = 0.4$ $Pr(Y=\gamma|X=c) = 0.8$

① Consider a simple detector that maps $a \to \alpha$, $b \to \beta$, $c \to \gamma$. What is the probability of error?

We know that the probability of error is 1 - probability of success. We can use the total probability theorem to calculate the probability of success:

$$Pr(X=Y) = Pr(X=a)Pr(Y=\alpha|X=a) + Pr(X=b)Pr(Y=\beta|X=b) + Pr(X=c)Pr(Y=\gamma|X=c)$$
$$= 0.3\cdot 0.6 + 0.5\cdot 0.5 + 0.2\cdot 0.8 = 0.59 \qquad 1 - 0.59 = 0.41$$

② What is the optimal detector?

First, assume $Y=\alpha$. Given $Y=\alpha$, we want to find the highest probability for X. We use Bayes' theorem 3 times. Omitted for space

③ What is the probability of error in the optimal detector?

Repeat the steps in part ①, but for the new detector

④ Does the probability of each value of X affect the detector?

Yes. The posterior probability is calculated based on the priori probability and transition probability. So the probability of transmission will affect the detector design. (i.e. if we increase $Pr(X=c)$ the optimal detector changes)

Random Variables/PMF/Expectation

5 balls in a bag labelled 1, 2, 3, 4, 5. Take 3 balls randomly. Let X denote the largest # on a ball picked (i.e. $X: \{1,2,3\} \mapsto 3$)

① Find the PMF of X

Support of X is 3, 4, 5. $\binom{5}{3} = \frac{5\cdot4\cdot\cancel{3}\cdot\cancel{2}}{\cancel{3}\cdot2\cdot\cancel{2}} = \frac{20}{2} = 10$ ways to pick 3 balls.

$X=3$ if $\{1,2,3\}$

$X=4$ in $\binom{3}{2} = \frac{3\cdot\cancel{2}}{\cancel{2}} = 3$ ways (ball 4 must be picked, 2 "free" balls)

$X=5$ in $\binom{4}{2} = \frac{4\cdot3\cdot\cancel{2}}{2\cdot\cancel{2}} = 6$ ways (ball 5 must be picked, 2 "free" balls)

Therefore: $p_X(x) = \begin{cases} .1 & \text{if } x=3 \\ .3 & \text{if } x=4 \\ .6 & \text{if } x=5 \end{cases}$

② Calculate $E[X]$

$$E[X] = (3)(.1) + (4)(.3) + (5)(.6) = .3 + 1.2 + 3 = 4.5$$

③ Given $X=4$, what is the probability you picked ball #1

3 ways for $X=4$ $\{\{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$. $\frac{2}{3}$ by discrete uniform law.

you got this