## Vectors

A vector is a set of coordinates. Notation: $\mathbf{v}$ or $\vec{v}$.

$$\vec{v} = \langle 1, 2, 3 \rangle$$

$$\vec{w} = \langle 1, 2 \rangle$$

Here, $\vec{v}$ is a vector in $\mathbb{R}^3$, and $\vec{w}$ is a vector in $\mathbb{R}^2$
The magnitude, or norm of a vector, represented by $||\vec{v}||$, is defined as $\sqrt{v_x^2 + v_y^2}$ in 2-space or $\sqrt{v_x^2 + v_y^2 + v_z^2}$ in 3-space.

## Vector addition and subtraction

We can add vectors component-wise:

$$\begin{aligned} \vec{v} &= \langle 1, 2, 3 \rangle \\ \vec{w} &= \langle 4, 5, 6 \rangle \\ \vec{v} + \vec{w} &= \langle 5, 7, 9 \rangle \end{aligned}$$

We can also subtract vectors component-wise:

$$\begin{aligned} \vec{v} &= \langle 4, 5, 6 \rangle \\ \vec{w} &= \langle 1, 2, 3 \rangle \\ \vec{v} - \vec{w} &= \langle 4-1, 5-2, 6-3 \rangle = \langle 3, 3, 3 \rangle \end{aligned}$$

Geometrically, vector addition works by putting vectors "tip to tail."

## Unit vectors

Vectors are often defined in terms of *unit vectors*:
**In $\mathbb{R}^2$:**

$$\begin{aligned} \hat{i} &= \langle 1, 0 \rangle \\ \hat{j} &= \langle 0, 1 \rangle \end{aligned}$$

**In $\mathbb{R}^3$:**

$$\begin{aligned} \hat{i} &= \langle 1, 0, 0 \rangle \\ \hat{j} &= \langle 0, 1, 0 \rangle \\ \hat{k} &= \langle 0, 0, 1 \rangle \end{aligned}$$

For example:

$$\langle 1, 2, 3 \rangle = \hat{i} + 2\hat{j} + 3\hat{k}$$

## Scalar multiplication

Vectors can be multiplied by scalars component-wise:

$$\lambda \langle a, b, c \rangle = \langle \lambda a, \lambda b, \lambda c \rangle$$

## Dot products

Taking the dot product is a method of multiplying vectors to produce a scalar. The formula for a dot product is

$$\langle a, b \rangle \cdot \langle x, y \rangle = ax + by$$
$$\langle a, b, c \rangle \cdot \langle x, y, z \rangle = ax + by + zc$$

Another way to write this is:

$$\vec{a} \cdot \vec{b} = ||\vec{a}||\ ||\vec{b}|| \cos(\theta)$$

Where $\theta$ is the angle between the vectors.
The dot product geometrically represents the scalar projection of one vector onto another.

## Cross products

Taking the cross product is a method of multiplying vectors to produce a vector. The formula for a cross product is:

$$\langle a, b, c \rangle \times \langle x, y, z \rangle = \langle -cy + bz, cx - az, -bx + ay \rangle$$

**Cross products are non-commutative**. Order does matter. $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (except in some very specific circumstances).
Cross products geometrically:

- From a right hand system (i.e. $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right hand system)
- Are orthogonal to the crossed vectors
- Have magnitude equal to the parallelogram spanned by the crossed vectors

## Lines

Lines are defined in terms of a point and a direction.

## Planes

Planes are defined in terms of three non-colinear points, or a normal vector and a plane.
To get a normal vector from 3 points, $A$, $B$, and $C$, compute $\vec{AB} \times \vec{AC}$
With normal vector $\vec{n}$ and point $P$:

$$\vec{n}_x(x - P_x) + \vec{n}_y(y - P_y) + \vec{n}_z(z - P_z) = 0$$

## Polar Coordinates

Polar coordinates are used to represent points in $\mathbb{R}^2$. They are represented as $(r, \theta)$, where $r \in \mathbb{R}$ and $\theta \in [0, 2\pi)$.
To convert between Cartesian coordinates and polar coordinates:

$$r = \sqrt{x^2 + y^2} \qquad x = r\cos\theta$$
$$\theta = \arctan\left(\frac{y}{x}\right) \qquad y = r\sin\theta$$

Mathematica snippet: `AngleVector[{x, y}]` will convert polar to rectangular.

## Spherical Coordinates

Spherical coordinates are one of two generalizations to $\mathbb{R}^3$ of polar coordinates. They are represented as $(\rho, \theta, \phi)$, where $\rho \in \mathbb{R}$, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$. $\rho$ represents the distance to the origin, $\theta$ represents the counterclockwise angle towards positive $y$ in the $xz$-plane, and $\phi$ represents the angle towards positive $x$ in the $xy$-plane.

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad x = \rho \sin\phi \cos\theta$$

$$\theta = \arctan\left(\frac{y}{x}\right) \qquad y = \rho \sin\phi \sin\theta$$

$$\rho = \arccos\left(\frac{z}{\rho}\right) \qquad z = \rho \cos\phi$$

## Cylindrical Coordinates

Cylindrical coordinates are one of two generalizations to $\mathbb{R}^3$ of polar coordinates. They are represented as $(r, \theta, z)$, where $r \in \mathbb{R}$, $\theta \in [0, 2\pi)$, and $z \in \mathbb{R}$. $r$ represents the distance to the origin, $\theta$ represents the counterclockwise angle towards positive $y$ in the $xz$-plane, and $z$ represents the distance from the $xy$-plane in the positive $z$ direction.

$$r = \sqrt{x^2 + y^2 + z^2} \qquad x = r \cos\theta$$

$$\theta = \arctan\left(\frac{y}{x}\right) \qquad y = r \sin\theta$$

$$z = z \qquad z = z$$

## Surfaces to remember

### Cylindrical Coordinates

| equation | description |
|---|---|
| $r = R$ | cylinder of radius $R$ |
| $\theta = \theta_0$ | vertical half-plane |
| $z = c$ | horizontal plane |

### Spherical Coordinates

| equation | description |
|---|---|
| $\rho = R$ | sphere of radius $R$ |
| $\theta = \theta_0$ | vertical half-plane |
| $\phi = \phi_0$ | right circular cone |

### Rectangular Coordinates

| equation | description |
|---|---|
| $x^2 + y^2 = z^2$ | right circular cone |
| $x^2 + y^2 + z^2 = R$ | sphere (radius $R$) |
| $x^2 + y^2 = R$ | cylinder (radius $R$) |

## Calculus of Vector-Valued Functions

Calculus can be done on vector-valued functions component-wise. This includes limits, differentiation, and integration. There are some additional differentiation rules

- Sum rule: $(\vec{r}_1(t) + \vec{r}_2(t))' = \vec{r}_1'(t) + \vec{r}_2'(t)$
- Chain rule: $\vec{r}\,(g(t)) = g'(t)\vec{r}'(g(t))$
- Product rules
  - Scalar product rule: $(\lambda\vec{r}(t))' = \lambda\vec{r}'(t)$
  - Dot product rule: $(\vec{r}_1 \cdot \vec{r}_2)' = \vec{r}_1' \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_2'$
  - Cross product rule: $(\vec{r}_1 \times \vec{r}_2)' = \vec{r}_1' \times \vec{r}_2 + \vec{r}_1 \times \vec{r}_2'$

    Remember! Cross products are non-commutative.

The derivative of a vector is also called the *tangent vector*, or *velocity vector*. This is because if $\vec{r}'(t_0)$ is non-zero, it points in the direction tangent to the curve at $r(t_0)$. The tangent line has parametrization:

$$\vec{L}(t) = \vec{r}(t_0) + t\vec{r}'(t_0)$$

## Arc length

If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ directly traverses curve $\ell$, for $a \leq t \leq b$, the arc length, $s$ of $\ell$ is:

$$\int_a^b ||\vec{r}'(t)|| = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

## Speed

The velocity vector, $\vec{v}$, points in the direction of travel. It's magnitude is the speed:

$$v(t) = \frac{ds}{dt} = ||\vec{r}'(t)||$$