Vectors

A vector is a set of coordinates. Notation: $\mathbf{v}$ or $\vec{v}$.

$$\vec{v} = \langle 1, 2, 3 \rangle$$

$$\vec{w} = \langle 1, 2 \rangle$$

Here, $\vec{v}$ is a vector in $\mathbb{R}^3$, and $\vec{w}$ is a vector in $\mathbb{R}^2$
The magnitude, or norm of a vector, represented by $||\vec{v}||$, is defined as $\sqrt{v_x^2 + v_y^2}$ in 2-space or $\sqrt{v_x^2 + v_y^2 + v_z^2}$ in 3-space.

Vector addition and subtraction

We can add vectors component-wise:

$$\vec{v} = \langle 1, 2, 3 \rangle$$
$$\vec{w} = \langle 4, 5, 6 \rangle$$
$$\vec{v} + \vec{w} = \langle 5, 7, 9 \rangle$$

We can also subtract vectors component-wise:

$$\vec{v} = \langle 4, 5, 6 \rangle$$
$$\vec{w} = \langle 1, 2, 3 \rangle$$
$$\vec{v} - \vec{w} = \langle 4 - 1, 5 - 2, 6 - 3 \rangle = \langle 3, 3, 3 \rangle$$

Geometrically, vector addition works by putting vectors "tip to tail."

Unit vectors

Vectors are often defined in terms of *unit vectors*:
In $\mathbb{R}^2$:

$$\hat{i} = \langle 1, 0 \rangle$$
$$\hat{j} = \langle 0, 1 \rangle$$

In $\mathbb{R}^3$:

$$\hat{i} = \langle 1, 0, 0 \rangle$$
$$\hat{j} = \langle 0, 1, 0 \rangle$$
$$\hat{k} = \langle 0, 0, 1 \rangle$$

For example:

$$\langle 1, 2, 3 \rangle = \hat{i} + 2\hat{j} + 3\hat{k}$$

Scalar multiplication

Vectors can be multiplied by scalars component-wise:

$$\lambda \langle a, b, c \rangle = \langle \lambda a, \lambda b, \lambda c \rangle$$

Dot products

Taking the dot product is a method of multiplying vectors to produce a scalar. The formula for a dot product is

$$\langle a, b \rangle \cdot \langle x, y \rangle = ax + by$$
$$\langle a, b, c \rangle \cdot \langle x, y, z \rangle = ax + by + zc$$

Another way to write this is:

$$\vec{a} \cdot \vec{b} = ||\vec{a}||\ ||\vec{b}|| \cos(\theta)$$

Where θ is the angle between the vectors.
The dot product geometrically represents the scalar projection of one vector onto another.

Cross products

Taking the cross product is a method of multiplying vectors to produce a vector. The formula for a cross product is:

$$\langle a, b, c \rangle \times \langle x, y, z \rangle = \langle -cy + bz, cx - az, -bx + ay \rangle$$

Cross products are non-commutative. Order does matter. $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (except in some very specific circumstances).
Cross products geometrically:

- From a right hand system (i.e. $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right hand system)
- Are orthogonal to the crossed vectors
- Have magnitude equal to the parallelogram spanned by the crossed vectors

Cross products always have the magnitude:

$$||a \times b|| = ||a||\ ||b|| \sin(\theta)$$

Lines

Lines are defined in terms of a point and a direction.

Planes

Planes are defined in terms of three non-colinear points, or a normal vector and a plane.
To get a normal vector from 3 points, A, B, and C, compute $\vec{AB} \times \vec{AC}$
With normal vector $\vec{n}$ and point P:

$$\vec{n}_x(x - P_x) + \vec{n}_y(y - P_y) + \vec{n}_z(z - P_z) = 0$$

Polar Coordinates

Polar coordinates are used to represent points in $\mathbb{R}^2$. They are represented as (r, θ), where $r \in \mathbb{R}$ and $\theta \in [0, 2\pi)$.

To convert between Cartesian coordinates and polar coordinates:

$$r = \sqrt{x^2 + y^2} \qquad x = r\cos\theta$$
$$\theta = \arctan\left(\frac{y}{x}\right) \qquad y = r\sin\theta$$

Mathematica snippet: `AngleVector[{x, y}]` will convert polar to rectangular.

Spherical Coordinates

Spherical coordinates are one of two generalizations to $\mathbb{R}^3$ of polar coordinates. They are represented as (ρ, θ, ϕ), where $\rho \in \mathbb{R}$, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$. ρ represents the distance to the origin, θ represents the counterclockwise angle towards positive y in the xz-plane, and ϕ represents the angle towards positive x in the xy-plane.

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad x = \rho\sin\phi\cos\theta$$
$$\theta = \arctan\left(\frac{y}{x}\right) \qquad y = \rho\sin\phi\sin\theta$$
$$\rho = \arccos\left(\frac{z}{\rho}\right) \qquad z = \rho\cos\phi$$

Cylindrical Coordinates

Cylindrical coordinates are one of two generalizations to $\mathbb{R}^3$ of polar coordinates. They are represented as (r, θ, z), where $r \in \mathbb{R}$, $\theta \in [0, 2\pi)$, and $z \in \mathbb{R}$. r represents the distance to the origin, θ represents the counterclockwise angle towards positive y in the xz-plane, and z represents the distance from the xy-plane in the positive z direction.

$$r = \sqrt{x^2 + y^2 + z^2} \qquad x = r\cos\theta$$
$$\theta = \arctan\left(\frac{y}{x}\right) \qquad y = r\sin\theta$$
$$z = z \qquad z = z$$

Surfaces to remember

Cylindrical Coordinates

equation	description
$r = R$	cylinder of radius R
$\theta = \theta_0$	vertical half-plane
$z = c$	horizontal plane

Spherical Coordinates

equation	description
$\rho = R$	sphere of radius R
$\theta = \theta_0$	vertical half-plane
$\phi = \phi_0$	right circular cone

Rectangular Coordinates

equation	description
$x^2 + y^2 = z^2$	right circular cone
$x^2 + y^2 + z^2 = R$	sphere (radius R)
$x^2 + y^2 = R$	cylinder (radius R)

Calculus of Vector-Valued Functions

Calculus can be done on vector-valued functions component-wise. This includes limits, differentiation, and integration. There are some additional differentiation rules

- Sum rule: $(\vec{r}_1(t) + \vec{r}_2(t))' = \vec{r}_1'(t) + \vec{r}_2'(t)$
- Chain rule: $\vec{r}\,(g(t)) = g'(t)\vec{r}'(g(t))$
- Product rules
  - Scalar product rule: $(\lambda\vec{r}(t))' = \lambda\vec{r}'(t)$
  - Dot product rule: $(\vec{r}_1 \cdot \vec{r}_2)' = \vec{r}_1' \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_2'$
  - Cross product rule: $(\vec{r}_1 \times \vec{r}_2)' = \vec{r}_1' \times \vec{r}_2 + \vec{r}_1 \times \vec{r}_2'$
    Remember! Cross products are non-commutative.

The derivative of a vector is also called the *tangent vector*, or *velocity vector*. This is because if $\vec{r}'(t_0)$ is non-zero, it points in the direction tangent to the curve at $r(t_0)$. The tangent line has parametrization:

$$\vec{L}(t) = \vec{r}(t_0) + t\vec{r}'(t_0)$$

Arc length

If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ directly traverses curve ℓ, for $a \le t \le b$, the arc length, s of ℓ is:

$$\int_a^b ||\vec{r}'(t)|| = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Speed

The velocity vector, $\vec{v}$, points in the direction of travel. It's magnitude is the speed:

$$v(t) = \frac{ds}{dt} = ||\vec{r}'(t)||$$