Vectors

A vector is a set of coordinates. Notation: $\mathbf{v}$ or $\vec{v}$.

$$\vec{v} = \langle 1, 2, 3\rangle$$

$$\vec{w} = \langle 1, 2\rangle$$

Here, $\vec{v}$ is a vector in $\mathbb{R}^3$, and $\vec{w}$ is a vector in $\mathbb{R}^2$
The magnitude, or norm of a vector, represented by $||\vec{v}||$, is defined as $\sqrt{v_x^2 + v_y^2}$ in 2-space or $\sqrt{v_x^2 + v_y^2 + v_z^2}$ in 3-space.

Vector addition and subtraction

We can add vectors component-wise:

$$\vec{v} = \langle 1, 2, 3\rangle$$
$$\vec{w} = \langle 4, 5, 6\rangle$$
$$\vec{v} + \vec{w} = \langle 5, 7, 9\rangle$$

We can also subtract vectors component-wise:

$$\vec{v} = \langle 4, 5, 6\rangle$$
$$\vec{w} = \langle 1, 2, 3\rangle$$
$$\vec{v} - \vec{w} = \langle 4-1, 5-2, 6-3\rangle = \langle 3, 3, 3\rangle$$

Geometrically, vector addition works by putting vectors "tip to tail."

Unit vectors

Vectors are often defined in terms of *unit vectors*:
In $\mathbb{R}^2$:

$$\hat{i} = \langle 1, 0\rangle$$
$$\hat{j} = \langle 0, 1\rangle$$

In $\mathbb{R}^3$:

$$\hat{i} = \langle 1, 0, 0\rangle$$
$$\hat{j} = \langle 0, 1, 0\rangle$$
$$\hat{k} = \langle 0, 0, 1\rangle$$

For example:

$$\langle 1, 2, 3\rangle = \hat{i} + 2\hat{j} + 3\hat{k}$$

Scalar multiplication

Vectors can be multiplied by scalars component-wise:

$$\lambda\langle a, b, c\rangle = \langle \lambda a, \lambda b, \lambda c\rangle$$

Dot products

Taking the dot product is a method of multiplying vectors to produce a scalar. The formula for a dot product is

$$\langle a, b\rangle \cdot \langle x, y\rangle = ax + by$$
$$\langle a, b, c\rangle \cdot \langle x, y, z\rangle = ax + by + zc$$

Another way to write this is:

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \left|\left|\vec{b}\right|\right| \cos(\theta)$$

Where θ is the angle between the vectors.
The dot product geometrically represents the scalar projection of one vector onto another.

Cross products

Taking the cross product is a method of multiplying vectors to produce a vector. The formula for a cross product is:

$$\langle a, b, c\rangle \times \langle x, y, z\rangle = \langle -cy + bz, cx - az, -bx + ay\rangle$$

Cross products are non-commutative. Order does matter. $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ (except in some very specific circumstances).
Cross products geometrically:

- From a right hand system (i.e. $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right hand system)
- Are orthogonal to the crossed vectors
- Have magnitude equal to the parallelogram spanned by the crossed vectors

Cross products always have the magnitude:

$$||a \times b|| = ||a||\ ||b|| \sin(\theta)$$

Parallelepipeds

A *parallelepiped* is the shape spanned by three non-zero vectors. Given vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, the base of the parallelepiped is the parallelogram spanned by $\vec{v}$ and $\vec{w}$. Therefore, the area of the base is $||\vec{v} \times \vec{w}||$. The volume of a parallelepiped $\mathscr{D}$ spanned by $\vec{u}$, $\vec{v}$ and $\vec{w}$ can be calculated with the *scalar triple product*:

$$V(\mathscr{D}) = |\vec{u} \cdot (\vec{v} \times \vec{w})| = \det(u, v, w)$$

Lines

Lines are defined in terms of a point and a direction, or two points. Given a direction vector (parallel to the line), $\vec{d}$, and a point on the line $\vec{p}$, the parameterization of the line is:

$$\vec{r}(t) = \vec{p} + t\vec{d}$$

Given two points, $\vec{p_1}$ and $\vec{p_2}$:

$$\vec{r}(t) = \vec{p_1} + t(\vec{p_2} - \vec{p_1})$$

Planes

Planes are defined in terms of three non-colinear points, or a normal vector and a plane.
To get a normal vector from 3 points, A, B, and C, compute $\vec{AB} \times \vec{AC}$
With normal vector $\vec{n}$ and point P:

$$\vec{n}_x(x - P_x) + \vec{n}_y(y - P_y) + \vec{n}_z(z - P_z) = 0$$

Polar Coordinates

Polar coordinates are used to represent points in $\mathbb{R}^2$. They are represented as (r, θ), where $r \in \mathbb{R}$ and $\theta \in [0, 2\pi)$.
To convert between Cartesian coordinates and polar coordinates:

$$r = \sqrt{x^2 + y^2} \qquad x = r\cos\theta$$
$$\theta = \arctan\left(\frac{y}{x}\right) \qquad y = r\sin\theta$$

Mathematica snippet: `AngleVector[{x, y}]` will convert polar to rectangular.

Spherical Coordinates

Spherical coordinates are one of two generalizations to $\mathbb{R}^3$ of polar coordinates. They are represented as (ρ, θ, ϕ), where $\rho \in \mathbb{R}$, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$. ρ represents the distance to the origin, θ represents the counterclockwise angle towards positive y in the xz-plane, and ϕ represents the angle towards positive x in the xy-plane.

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad x = \rho\sin\phi\cos\theta$$
$$\theta = \arctan\left(\frac{y}{x}\right) \qquad y = \rho\sin\phi\sin\theta$$
$$\rho = \arccos\left(\frac{z}{\rho}\right) \qquad z = \rho\cos\phi$$

Cylindrical Coordinates

Cylindrical coordinates are one of two generalizations to $\mathbb{R}^3$ of polar coordinates. They are represented as (r, θ, z), where $r \in \mathbb{R}$, $\theta \in [0, 2\pi)$, and $z \in \mathbb{R}$. r represents the distance to the origin, θ represents the counterclockwise angle towards positive y in the xz-plane, and z represents the distance from the xy-plane in the positive z direction.

$$r = \sqrt{x^2 + y^2 + z^2} \qquad x = r\cos\theta$$
$$\theta = \arctan\left(\frac{y}{x}\right) \qquad y = r\sin\theta$$
$$z = z \qquad z = z$$

Surfaces to remember

Cylindrical Coordinates:	
equation	description
$r = R$	cylinder of radius R
$\theta = \theta_0$	vertical half-plane
$z = c$	horizontal plane
Spherical Coordinates:	
equation	description
$\rho = R$	sphere of radius R
$\theta = \theta_0$	vertical half-plane
$\phi = \phi_0$	right circular cone
Rectangular Coordinates:	
equation	description
$x^2 + y^2 = z^2$	right circular cone
$x^2 + y^2 + z^2 = R$	sphere (radius R)
$x^2 + y^2 = R$	cylinder (radius R)

Calculus of Vector-Valued Functions

Calculus can be done on vector-valued functions component-wise. This includes limits, differentiation, and integration. There are some additional differentiation rules

- Sum rule: $(\vec{r}_1(t) + \vec{r}_2(t))' = \vec{r}_1'(t) + \vec{r}_2'(t)$
- Chain rule: $\vec{r}(g(t)) = g'(t)\vec{r}'(g(t))$
- Product rules
  - Scalar product rule: $(\lambda\vec{r}(t))' = \lambda\vec{r}'(t)$
  - Dot product rule: $(\vec{r}_1 \cdot \vec{r}_2)' = \vec{r}_1' \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_2'$
  - Cross product rule: $(\vec{r}_1 \times \vec{r}_2)' = \vec{r}_1' \times \vec{r}_2 + \vec{r}_1 \times \vec{r}_2'$
    Remember! Cross products are non-commutative.

The derivative of a vector is also called the *tangent vector*, or *velocity vector*. This is because if $\vec{r}'(t_0)$ is non-zero, it points in the direction tangent to the curve at $r(t_0)$. The tangent line has parametrization:

$$\vec{L}(t) = \vec{r}(t_0) + t\vec{r}'(t_0)$$

Arc length

If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ directly traverses curve $\mathscr{L}$, for $a \le t \le b$, the arc length, s of $\mathscr{L}$ is:

$$\int_a^b ||\vec{r}'(t)|| = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Speed

The velocity vector, $\vec{v}$, points in the direction of travel. It's magnitude is the speed:

$$v(t) = \frac{ds}{dt} = ||\vec{r}'(t)||$$

Functions in Multiple Variables

A function can exist in *multiple variables* if it takes several parameters.
If $f(x,y)$ is defined near $P=(a,b)$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if, $\forall\epsilon > 0\ \exists\delta > 0$ s.t. $0 < d((x,y),(a,b)) < \delta$ then $|f(x,y)-L| < \epsilon$.
A function in two variables is *continuous* at (a,b) if the following equation holds:

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Partial Derivatives

The partial derivatives of $f(x,y)$ are defined as the limits:

$$f_x(a,b) = \left[\frac{\delta f}{\delta x}\right]_{(a,b)} = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$
$$f_y(a,b) = \left[\frac{\delta f}{\delta y}\right]_{(a,b)} = \lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{h}$$

Partial derivatives can be approximated for small Δx and Δy values:

$$f_x(a,b) \approx \frac{\Delta f}{\Delta x} = \frac{f(a+\Delta x,b)-f(a,b)}{\Delta x}$$
$$f_y(a,b) \approx \frac{\Delta f}{\Delta y} = \frac{f(a,b+\Delta y)-f(a,b)}{\Delta y}$$

The second-order partial derivatives are written as:

$$\frac{\delta^2}{\delta x^2}f = f_{xx} \qquad \frac{\delta^2}{\delta x\delta y} = f_{xy} \qquad \frac{\delta^2}{\delta y^2}f = f_{yy}$$

Clairaut's Theorem

Clairaut's theorem states that mixed partials are equivalent as long as they are continuous: $f_{xy} = f_{yx}$.

Differentiability of Functions in Multiple Variables

$f(x,y)$ is *differentiable* at (a,b) if $f_x(a,b)$ and $f_y(a,b)$ both exist, and

$$\lim_{(x,y)\to(a,b)}\frac{f(x,y)-L(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

where

$$L(x,y) = f(a,b)+f_x(a,b)\times(x-a)+f_y(a,b)\times(y-b)$$

Gradient Vectors

The gradient vector is defined is

$$\mathbb{R}^2 : \nabla f = \left\langle \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}\right\rangle$$
$$\mathbb{R}^3 : \nabla f = \left\langle \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z}\right\rangle$$

This gradient has the properties:

- ∇f_P points in the fastest rate of increase, and the rate of that increase is $||\nabla f_P||$
- $-\nabla f_P$ points in the fastest rate of decrease, and the rate of that decrease is $-||\nabla f_P||$
- $-\nabla f_P$ is orthogonal to level curves through P

The chain rule for paths also applies:

$$\frac{d}{dt}f(\vec{r}(t)) = \nabla f_{r(t)}\cdot\vec{r'}(t)$$

Directional Derivatives

For unit vector $\vec{u} = \langle h,k\rangle$, $D_u f(a,b)$ is the directional derivative with respect to $\vec{u}$.

$$D_u f(a,b) = \lim_{t\to 0}\frac{f(a+th,b+tk)-f(a,b)}{t}$$

For a differentiable function f, the directional derivative in the direction $\vec{u}$ is computed using the gradient:

$$D_u f(a,b) = \nabla f_{(a,b)}\cdot\vec{u} = ||\nabla f_{(a,b)}||\cos\theta$$

Tangent Plane

The equation for a tangent plane can be thought of as the equation for a tangent line generalized into three dimensions. The equation for a tangent line at x_0 is (derived from point-slope form):

$$y = f(x_0)+f'(x_0)\times(x-x_0)$$

The equation for a tangent plane of $f(x,y)$ at the point (a,b) is:

$$z = f(a,b)+f_x(a,b)\times(x-a)+f_y(a,b)\times(y-b)$$

A normal vector can be computed using this tangent plane equation.

Linear Approximation

If $f(x,y)$ is differentiable at (a,b), the *linearization* of f centered at (a,b) is:

$$L(x,y) = f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

You can use this linearization to compute *linear approximations* of functions:

$$f(a+\Delta x,b+\Delta y)\approx f(a,b)+f_x(a,b)\Delta x+f_y(a,b)\Delta y$$
$$\Delta f \approx f_x(a,b)\Delta x+f_y(a,b)\Delta y$$

This can be expressed in a differential form as well:

$$df = f_x(x,y)dx+f_y(x,y)dy = \frac{\delta f}{\delta x}dx+\frac{\delta f}{\delta y}dy$$

Chain rules

If $f(x, y, z)$ is a composite function (meaning that it can be represented in a form similar to $f(x(s, t), y(s, t), z(s, t))$, where s, and t are the *independent variables*), then

$$\frac{\delta f}{\delta s} = \frac{\delta f}{\delta x}\frac{\delta x}{\delta s} + \frac{\delta f}{\delta y}\frac{\delta y}{\delta s} + \frac{\delta f}{\delta z}\frac{\delta z}{\delta s}$$
$$\frac{\delta f}{\delta t} = \frac{\delta f}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta f}{\delta y}\frac{\delta y}{\delta t} + \frac{\delta f}{\delta z}\frac{\delta z}{\delta t}$$

Critical Points

$P = (a, b)$ is a *critical point* if $f_x(a, b) = 0$ or $f_x(a, b)$ does not exist, and $f_y(a, b) = 0$ or $f_y(a, b)$ does not exist.
The *discriminant* of $f(x, y)$ at $P = (a, b)$ is defined as the quantity:

$$D(a, b) = f_{xx}(a, b) \times f_{yy}(a, b) - f_{xy}^2(a, b)$$

The *second derivative test* allows us to categorize critical points. Give a critical point, (a, b) on f:

$$\begin{aligned} D(a,b) > 0,\ f_{xx}(a,b) > 0 &\implies \text{local minimum} \\ D(a,b) > 0,\ f_{xx}(a,b) < 0 &\implies \text{local maximum} \\ D(a,b) < 0 &\implies \text{saddle point} \\ D(a,b) = 0 &\implies \text{test inconclusive} \end{aligned}$$

To find extreme values, first find the values at the critical points, then compare to the maximum value along the boundary.

Lagrange Multipliers

If the local extreme of $f(x, y)$ is subject to a constraint, $g(x, y) = 0$, then we can say that the critical points satisfy the *Lagrange condition* $\nabla f_P = \lambda \nabla g_P$. This is equivalent to the *Lagrange equations*:

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$

For functions in 3 variables, use 2 constraints: $g(x, y, z) = 0$ and $h(x, yz) = 0$, then use two Lagrange multipliers:

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

Multiple integration

Multiple integration can be computed in any order, for example:

$$\iint f(x, y)\, dx\, dy = \iint f(x, y)\, dy\, dx$$

Just integrate them as if they were single integrals, multiple times. This property is known as *Fubini's theorem*.

Riemann Sums in Multiple Variables

A *Riemann sum* for $f(x, y)$ on $\mathscr{R} = [a, b] \times [c, d]$ is a sum in the form:

$$S_{N,M} = \sum_{i=1}^{N}\sum_{j=1}^{M} f(P_{ij})\Delta x_i \Delta y_j$$

Double integrals over complex regions

Assume that $\mathscr{D}$ is a closed, bounded domain whose boundary is a simple closed curve that is either smooth or has a finite number of corners. The double integral is defined by:

$$\iint_{\mathscr{D}} f(x, y) dA = \iint_{\mathscr{R}} \tilde{f}(x, y) dA$$

where $\tilde{f}(x, y) = f(x, y)$ if $(x, y) \in \mathscr{D}$, otherwise $\tilde{f}(x, y) = 0$.
If $\mathscr{D}$ is vertically simple, meaning $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, then evaluate:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

If $\mathscr{D}$ is horizontally simple, meaning $h_1(y) \leq x \leq h_2(y)$ and $a < y < b$, then evaluate:

$$\int_a^b \int_{h_1(x)}^{h_2(x)} f(x, y)\, dx\, dy$$

Properties of Multiple Integration

If m is the minimum value and M is the maximum value of f on $\mathscr{D}$, then

$$m \cdot \text{area}(\mathscr{D}) \leq \iint_{\mathscr{D}} f(x, y)\, dA \leq M \cdot \text{area}(\mathscr{D})$$

If $z_1(x, y) \leq z_2(x, y)\ \forall (x, y) \in \mathscr{D}$, then the volume, V, of the solid region between the surfaces given by $z = z_1(x, y)$ and $z = z_2(x, y)$ over $\mathscr{D}$ is

$$V = \iint_{\mathscr{D}} (z_2(x, y) - z_1(x, y))\, dA$$

Mean value theorem

The mean value of f over $\mathscr{D}$, $\bar{f}$ is

$$\bar{f} = \frac{1}{\text{area}(\mathscr{D})} \iint_{\mathscr{D}} f(x, y)\, dA = \frac{\iint_{\mathscr{D}} f(x, y)\, dA}{\iint_{\mathscr{D}} 1\, dA}$$

If $f(x, y)$ is continuous and $\mathscr{D}$ is closed, bounded, and connected, there exists a point P such that $f(P) = \bar{f}$.